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An analytical relation between entropy production and quantum Lyapunov exponents for Gaussian bipartite systems

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Abstract
We study and compare the information loss of a large class of Gaussian bipartite systems. It includes the usual Caldeira–Leggett-type model as well as Anosov models (parametric oscillators, the inverted oscillator environment, etc), which exhibit instability, one of the most important characteristics of chaotic systems. We establish a rigorous connection between the quantum Lyapunov exponents and coherence loss, and show that in the case of unstable environments coherence loss is completely determined by the upper quantum Lyapunov exponent, a behavior which is more universal than that of the Caldeira–Leggett-type model.

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1. Introduction

The application of quantum mechanics to real-world many-body systems meets with several difficulties, of both conceptual and pragmatic nature. A quantum mechanical system, which consists of at least two interacting subsystems exhibits a completely nonclassical property called ‘entanglement’ [1]. This is a nonclassical correlation between systems which exists even between well-separated subsystems [2, 3]. This unique property of a quantum system is nowadays viewed as a powerful resource in quantum information theory and quantum computation [4]. Also, on the conceptual side, ‘entanglement’ with an environment (degrees of freedom which inevitably interact with the quantum system of interest) is the key mechanism to explain why typically quantum effects are not observed in macroscopic systems [5, 6].

Therefore, the rate at which pure initial quantum states lose their potentiality to retain information represents at the same time an old and a very modern problem.

The first theoretical implementation of the system plus environment dynamics, in both the weak and strong damping regimes, was proposed and carried out by Caldeira and Leggett [7]. They represent the environment as an infinity of oscillators, weakly coupled to the system of interest. Several successful descriptions of experiments, specifically in quantum optics, were based on this type of model. More recently questions have been raised about the role of classical chaos on the information loss process for coupled quantum systems. In particular, the role of chaos in the decoherence process [8] has become a matter of active research and also a matter of debate [9–11].

One of the important issues now is: which type of environment is the more effective—the one involving an infinite number of degrees of freedom (the quantum Brownian motion environment, QBME), as proposed by Caldeira and Leggett, or an environment consisting of a single degree of freedom which presents one or more characteristics of the chaotic behavior?

In the standard model of decoherence, a joint system consisting of a system (S) coupled to an environment (E) is described by a Hamiltonian

$$H_J = H_S + H_E + V, \tag{1.1}$$

where H_S is the system's Hamiltonian, H_E the environment's and V is the interaction Hamiltonian between S and E. Let, for simplicity, the state space $\mathcal{H}_S$ of the system be two-dimensional, and

$$H_S = \lambda\sigma_z, \tag{1.2}$$

with σ_z the Pauli matrix with eigenvalues +1 and −1 corresponding to spin 'up' and 'down' along the z-axis, and respective eigenfunctions ψ_+ and ψ_-. In the coherent superposition

$$\psi = \alpha_+\psi_+ + \alpha_-\psi_-, \tag{1.3}$$

with probability amplitudes α_+ and α_- such that $|\alpha_+|^2+|\alpha_-|^2 = 1$, σ_z has no definite value. The environment state (Hilbert) space is $\mathcal{H}_E$, and we suppose E is initially in the state $\psi_E(0) \in \mathcal{H}_E$ and the joint system in the pure (nonentangled) state

$$\psi_J(0) = \psi \otimes \psi_E(0) \in \mathcal{H}_S \otimes \mathcal{H}_E. \tag{1.4}$$

We also assume for simplicity that the interaction V in (1.1) is of the form

$$V = \mu\sigma_z Q_E. \tag{1.5}$$

Then, after a time t, the joint system will be in the state (units are chosen such that $\hbar = 1$)

$$\psi_J(t) = \big(\alpha_+ \exp[-i\lambda t]\psi_+ \otimes \psi_E^+(t) + \alpha_- \exp[i\lambda t]\psi_- \otimes \psi_E^-(t)\big), \tag{1.6a}$$

with

$$\psi_E^\pm(t) = \exp[-i(H_E \pm \mu Q_E)]\psi_E(0). \tag{1.6b}$$

We thus expect that the sensitivity to perturbation $\pm\mu Q_E$ of a chaotic environment Hamiltonian H_E will, by (1.6*a*) and (1.6*b*), result in quick decoherence:

$$\big|\big(\psi_E^+(t), \psi_E^-(t)\big)\big| \leqslant c \exp[-dt], \tag{1.7}$$

where d and c are positive constants. The exponential 'orbital' instability of H_E is expected to lead via (1.6*b*) to the exponential decay of the 'overlaps' (1.7) (think of Gaussians centered at the corresponding orbits): the decoherence rate should therefore be proportional to the rate at which the environment is able to explore its phase space. These considerations are meant to explain, intuitively, why a (large) collection of harmonic oscillators—being stable systems—could, *in principle*, be less effective for purposes of decoherence than one sole unstable system.

Recently, Blume-Kohout and Zurek [12] analyzed decoherence due to a toy model for the environment, an inverted harmonic oscillator environment (IHOE)—see also [13–15] where conclusions similar to ours were obtained in chaotic systems. This is a nontrivial analytical tractable model whose importance as realized in [12] is that it shares one very important characteristic with chaotic systems, namely exponential instability, defined by a positive upper Lyapunov exponent [16]. Systems of this sort are called Anosov systems. We shall be rather dealing with generalized Anosov systems, in the sense that the flows take place in a manifold which is not compact. A simple example is the flow in the (p, q) plane defined (with λ positive) by $\frac{\mathrm{d}p}{\mathrm{d}t} = \lambda p$ and $\frac{\mathrm{d}q}{\mathrm{d}t} = -\lambda q$. The q coordinates of any two points moving with the flow get closer and closer as time proceeds, but the p coordinates separate exponentially fast, and hence the two points move apart exponentially fast [17]. The directions of stable and unstable manifolds may vary from point to point, however, in contrast to the above example: this is the case of the parametric oscillator. The quantum analogues of the classical Anosov systems have been dubbed *quantum Anosov systems* [18]: they exhibit a dynamic behavior which is at the same time rich and universal and are characterized by having a positive upper quantum Lyapunov exponent, as defined in [19] and in (4.5), and will be the subject of the present paper.

Given the importance of the subject, the difficulty in obtaining mathematically sound exact results which are eventually able to shed light onto these complex questions, we feel that a thorough analytical investigation of the QBME versus IHOE and its generalizations to more realistic models of the class of linearly coupled quantum Anosov systems, can be extremely useful. We focus on the limited but important class of Gaussian states. In this context, a rigorous demonstration of the growth of the system's reduced von Neumann entropy with the upper quantum Lyapunov exponent for the aforementioned class of systems is given, showing thus that the rate of information loss is of more universal nature than that for the usual quantum Brownian motion environment. Our analytical results are only possible due to the fact that for (possibly nonunitary) Gaussian dynamics the rate of information loss is completely governed by a combination of quadratures, i.e. the covariance matrix, also called the Schrödinger generalized uncertainty principle.

2. Relation between information loss and the covariance matrix for the Gaussian states

We start by recalling a well-known result [20, 21]. The most general 1D Gaussian state can be written as

$$\hat{\rho}_G = \mathcal{D}(\alpha)\mathcal{S}(r,\phi)\hat{\rho}_\nu\mathcal{S}^\dagger(r,\phi)\mathcal{D}^\dagger(\alpha), \tag{2.1}$$

where

$$\mathcal{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}), \tag{2.2}$$

$$\mathcal{S}(r,\phi) = \exp\left(\frac{r}{2}(\mathrm{e}^{\mathrm{i}\phi}\hat{a}^{\dagger 2} - \mathrm{e}^{-\mathrm{i}\phi}\hat{a}^2)\right) \tag{2.3}$$

are the displacement and the squeezing operator, respectively, and $\hat{\rho}_\nu$ is the thermal density operator with average number of excitations ν,

$$\hat{\rho}_\nu = \frac{1}{1+\nu}\exp\left(\ln\left(\frac{\nu}{\nu+1}\right)\hat{a}^\dagger\hat{a}\right). \tag{2.4}$$

Note that except in the improper limit of $\nu \to 0$, which corresponds to a squeezed state, density matrix (2.1) is not a pure state. Its Schrödinger determinant is given by

$$D = \begin{pmatrix} \sigma_{pp} & \sigma_{qp} \\ \sigma_{qp} & \sigma_{qq} \end{pmatrix} = \left(\nu + \frac{1}{2}\right)^2, \tag{2.5}$$

where $\sigma_{xy} = \frac{1}{2}\operatorname{tr}(\hat\rho\{\hat x, \hat y\}) - \operatorname{tr}(\hat x\hat\rho)\operatorname{tr}(\hat y\hat\rho)$ and $\hat x$, $\hat y$ are either the position or its canonically conjugate momentum. We moreover have for the (von Neumann) entropy

$$S[\hat\rho] = -\operatorname{tr}(\hat\rho \ln \hat\rho) = (\nu+1)\ln(\nu+1) - \nu \ln \nu. \tag{2.6}$$

Comparing (2.6) with (2.5) we obtain the relation between the entropy and the Schrödinger determinant:

$$S = \left(\sqrt{D(t)} + \tfrac{1}{2}\right)\ln\left(\sqrt{D(t)} + \tfrac{1}{2}\right) - \left(\sqrt{D(t)} - \tfrac{1}{2}\right)\ln\left(\sqrt{D(t)} - \tfrac{1}{2}\right). \tag{2.7}$$

It is worthwhile to point out that (2.7) holds for a wide range of (Gaussian) states, ranging from thermal (with $\mathcal{S}(r,\phi) = 1, \mathcal{D}(\alpha) = 1$ in (2.1)) to generalized coherent states (the limit $\nu \to 0$ in (2.4), with $\mathcal{S} = 1$ in (2.1)); (2.7) is thus a fundamental relation for the Gaussian states.

3. Quantum Brownian motion environment (QBME)

We next consider an example of QBME widely used in the context of quantum optics, whose dynamics (in the Born–Markov approximation) is governed by the Liouvillian

$$\begin{aligned}\mathcal{L} = -\mathrm{i}\omega[\hat a^\dagger \hat a, \bullet] &+ k(\bar n_B + 1)(2\hat a \bullet \hat a^\dagger - \hat a^\dagger \hat a \bullet - \bullet\, \hat a^\dagger \hat a) \\ &+ k\bar n_B(2\hat a^\dagger \bullet \hat a - \hat a \hat a^\dagger \bullet - \bullet\, \hat a \hat a^\dagger).\end{aligned} \tag{3.1}$$

The entropy is a function of $\nu(t)$ only (see (2.6)), where

$$\nu(t) = \sqrt{(\sigma_{a^\dagger a}(t))^2 - \sigma_{a^\dagger a^\dagger}(t)\sigma_{aa}(t)} - \tfrac{1}{2}, \tag{3.2}$$

and

$$\sigma_{a^\dagger a}(t) = \left(\nu(0) + \tfrac{1}{2}\right)\cosh(2r(0))\,\mathrm{e}^{-2kt} + \left(\bar n_B + \tfrac{1}{2}\right)(1 - \mathrm{e}^{-2kt}), \tag{3.3}$$

$$\sigma_{a^\dagger a^\dagger}(t)\sigma_{aa}(t) = \mathrm{e}^{-4kt}\left(\left(\nu(0) + \tfrac{1}{2}\right)^2 \sinh^2(2r(0))\right). \tag{3.4}$$

Note that the entropy for this type of system may be obtained without resorting to the calculation of the full density matrix, due to the fact that the information exchange between the system and bath is completely determined by a simple combination of quadratures. Also the entropy saturates for long enough times. Moreover, from equation (3.2) is apparent that the information loss depends both on the environment constants as well as on the initial conditions.

4. Inverted harmonic environment (IHE)

We now revisit the inverted harmonic oscillator example. Following (1.1)), we have now

$$H_{\mathrm{S}} = p_1^2/2 + \omega_1^2 x_1^2/2, \tag{4.1}$$

$$H_{\mathrm{E}} = p_2^2/2 + \omega_2^2 x_2^2/2, \tag{4.2}$$

$$V = \lambda x_1 x_2. \tag{4.3}$$

The above Hamiltonian models two coupled harmonic oscillators: the first one, is a regular oscillator, with unit frequency, describing the system; the second one, an inverted oscillator with squared frequency $\omega_2^2 = -\Lambda^2$, describing the environment. This is a nontrivial tractable model. Its importance, as has been realized in [12], relies on its relation to chaos. Chaotic behavior requires besides stretching and folding in phase space, instabilities characterized by the positive Lyapunov exponents. Although the above Hamiltonian is defined on the whole of phase space and does not exhibit chaotic behavior, it displays positive Lyapunov exponents like all Anosov systems. In this section, we establish a connection between Lyapunov exponents and the rate of information loss, in this model, which we generalize to periodic quadratic Anosov systems in the following section and exemplify with a physically more sound system of coupled parametric oscillators. The solution of the Heisenberg equations of motion for the vector $\hat{\mathbf{z}} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2)^T$ is as follows [22]:

$$\hat{\mathbf{z}}(t) = \mathbf{G}\,\mathrm{e}^{\mathbf{L}t}\mathbf{G}^{(-1)}\hat{\mathbf{z}}, \tag{4.4}$$

where $\mathbf{G}$ and $\mathbf{L}$ are the constant matrices. The former is invertible and the latter is diagonal with elements $\pm\lambda = \pm A(\Lambda, \alpha)$ and $\pm\mu = \pm\mathrm{i}B(\Lambda, \alpha)$, where $A(\Lambda, \alpha)$ and $B(\Lambda, \alpha)$ are *real* and *positive* constants. The IHE Hamiltonian is a degenerate case of Floquet's theorem, see the following. Note the existence of an unstable mode, with the positive classical Lyapunov exponent λ. In [19], upper quantum Lyapunov exponents were defined, for systems of one degree of freedom with momentum $\hat{p}$ and position $\hat{x}$, as follows:

$$\bar{\lambda} = \sup_{\alpha\in\mathbf{R}^2} \limsup_{t\to\infty} \frac{1}{t} \ln\|[L_\alpha, A(t, t_0)]\|, \tag{4.5}$$

where $\alpha = (\alpha_p, \alpha_x)$ and $A(t, t_0)$ is any bounded operator, i.e. with the finite norm given by $\|A\| = \sup_\psi \|A\psi\|/\|\psi\|$, where ψ belongs to the Hilbert space $L_2(\mathbf{R}, \mathrm{d}x)$ and evolved in the Heisenberg picture. This norm is nothing but the natural extension of the usual matrix norm on the square-integrable sequences to an infinite-dimensional Hilbert space. The operator $L_\alpha = \alpha_p\hat{p} + \alpha_x\hat{x}$ is the generator of phase-space translations along a direction α, and thus (4.5) has a close resemblance to the definition of the classical Lyapunov exponents (see, e.g., [23]), but exploits the unitary nature of the quantum dynamics (see [19] for a discussion of these points).

Also in [19], definition (4.5) was applied for one of the simplest paradigms of the transition from regular to unstable behavior in classical mechanics, namely the parametrically driven oscillator, given by

$$H(t) = \tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}f(t)\hat{x}^2, \tag{4.6}$$

where $f(t)$ is a periodic function. The dynamical solution for $\hat{x}(t, 0)$ and $\hat{p}(t, 0)$ is a formally one-dimensional version of (4.4), but with $\mathbf{G}$ being a periodic matrix with the same period of the driving and $\mathbf{L}$ is a traceless matrix: this is Floquet's theorem and the eigenvalues of $\mathbf{L}$ are known as Floquet exponents (see [19] for the proof and references). Applying definition (4.5), it was found that the upper quantum Lyapunov exponent is precisely the real part of the maximal Floquet exponent, which is positive in the classical instability region of the parametric oscillator, and zero in the stability region. The generalization to two degrees of freedom is straightforward [24] and, applied to the IHE shows that the upper quantum Lyapunov exponent (4.5) is λ (see equation eigenvalues of the IHE). Let now $S_r = -\mathrm{tr}(\rho_1 \ln \rho_1)$, the von Neumann entropy of the reduced density matrix, with $\rho_1 \equiv tr_2\rho$, with ρ of the form (2.1)–(2.4), but now with $\nu \to 0$, i.e., the initial state being a tensor product of the generalized coherent states.

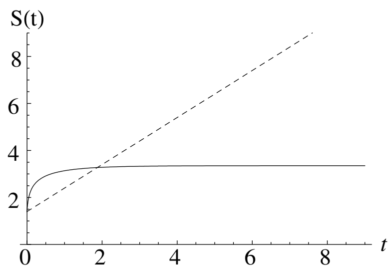

Figure 1. Time evolution of the entropy for a quantum Brownian motion environment (solid curve) and for an inverted harmonic oscillator reservoir (dash curve).

Since ρ_1 is also Gaussian, it continues to satisfy (2.7). In both cases, IHE and parametric oscillator (4.6), the determinant is found to be of the form

$$D(t) = \sum_{a,b=-2}^{2} C_{ab}\,\mathrm{e}^{(a\lambda+b\mu)t}, \tag{4.7}$$

where the constant coefficients C_{ab} are given by cumbersome not particularly enlightening expressions. These coefficients, which are zero unless the sum of a and b is an even number, depend on the variances of the initial state and will be omitted here. For instance, one can find C_{ab} by looking at the asymptotic behavior of (2.6). For large enough times t we have $D(t) \sim C_{20}\,\mathrm{e}^{2\lambda t}$ and the reduced entropy can be approximated by $S_r \sim \ln(\sqrt{D} - 1/2) + 1 \sim \ln D/2$. This behavior,

$$S_r(t) \sim \ln(C_{20})/2 + \lambda t, \tag{4.8}$$

is precisely the one conjectured by Zurek several years ago [12]. Since ρ_1 is also Gaussian, (2.7) applies to ρ_1 and thus the linear growth of $S_r(t)$ is interpreted, in the present case of a bipartite system, as a coherence loss, which, as we see, *depends only on the upper quantum Lyapunov exponent* (uqLe) and is thus universal for this class of systems, in contrast to QBME. Note that the complexity in the present models (they are not explicitly soluble classically and are a paradigm of the transition from regular to irregular behavior, see [25]) is brought about by the external field, which depends on time in a nonlinear way! For the models with frequency varying almost periodically with time—included in the present treatment—see [26]: it may be considered as a prototype of complexity. This is one of the reasons why our extension of [12] is significant: Zurek's IHOE is *not* complex!

As remarked in [12, section 2, p 032104–2], the IHE Hamiltonian is unphysical in that in general the directions of stable and unstable manifolds can vary from point to point. Thus, it is of special interest to be able to consider as system 2 an arbitrary quantum Anosov system. In what follows, we illustrate this by taking both systems of the form (4.6), a case in which most computations can be done explicitly. This model is particularly rich: there is even a transition from a stability to an instability region, which has a physical interpretation in a model of quadrupole radio-frequency traps (Paul-Penning traps) (see [19] and references therein).

For the sake of comparison, in figure 1 we plot the entropy obtained for a quantum Brownian motion environment (BME) and for a IHO reservoir (parameters: (i) for the BME $\nu_0 = 1, r_0 = 1, k = 0.5, n_B = 10, \omega = 1, \phi = 0$; (ii) for the IHO $\ln(C_{20})/2 = 1.4$ and $\lambda = 1$). Note that the increase in the entropy in the case of BME saturates when the system thermalizes with the reservoir, while the increase in the case of IHO is linear (in the asymptotic regime).

5. Coupled parametric oscillators

Consider now a more realistic example, physically relevant in connection with the Paul traps [27], which models two coupled parametric oscillators with Hamiltonian

$$H(t) = H_S(t) + H_E(t) + V, \tag{5.1}$$

where

$$H_S(t) = \tfrac{1}{2}\hat{p}_1^2 + \left(\omega_1^2 - q\cos 2t\right)\hat{x}_1^2, \tag{5.2}$$

$$H_E(t) = \tfrac{1}{2}\hat{p}_2^2 + \left(\omega_2^2 - q\cos 2t\right)\hat{x}_2^2 \tag{5.3}$$

and

$$V = \frac{g}{2}(\hat{x}_1 - \hat{x}_2)^2. \tag{5.4}$$

In what follows, we assume $g > 0$ and $\omega_2^2 > \omega_1^2$. We remark that (5.1) corresponds to the standard model (1.1). The solution of the Heisenberg equations of motion for the position and momentum operators can be written in terms of the Mathieu functions. For example, the momentum of the second oscillator, $\hat{p}_2(t)$, is given by the expression $u_1(t)\hat{x}_1(0) + u_2(t)\hat{p}_1(0) + u_3(t)\hat{x}_2(0) + u_4(t)\hat{p}_2(0)$, where the functions $u_a(t)$, $a = 1, 2, 3, 4$ are given, respectively, by

$$u_1 = \sin(2\theta)\left[\frac{\dot{S}(\alpha_-, q, 0)}{2D_2}C(\alpha_-, q, t) - \frac{\dot{S}(\alpha_+, q, 0)}{2D_1}C(\alpha_+, q, t)\right], \tag{5.5}$$

$$u_2 = \cos^2\theta\frac{\dot{S}(\alpha_-, q, 0)}{D_2}C(\alpha_-, q, t) - \sin^2\theta\frac{\dot{S}(\alpha_+, q, 0)}{D_1}C(\alpha_+, q, t), \tag{5.6}$$

$$u_3 = \sin(2\theta)\left[\frac{C(\alpha_-, q, 0)}{2D_2}S(\alpha_-, q, t) - \frac{C(\alpha_+, q, 0)}{2D_1}S(\alpha_+, q, t)\right], \tag{5.7}$$

$$u_4 = \cos^2\theta\frac{C(\alpha_-, q, 0)}{D_2}S(\alpha_-, q, t) - \sin^2\theta\frac{C(\alpha_+, q, 0)}{D_1}S(\alpha_+, q, t). \tag{5.8}$$

The parameters $\alpha_\pm = \frac{\omega_1^2+\omega_2^2}{2} + g \pm \sqrt{g^2 + \frac{(\omega_2^2-\omega_1^2)^2}{4}}$ depend on the constants appearing in the Hamiltonian. The angle θ appearing in the expressions above is determined from the equality: $\tan(2\theta) = \frac{2g}{\omega_2^2-\omega_1^2}$. The functions $C(\alpha, q, t)$ and $S(\alpha, q, t)$ are the usual Mathieu cosine and sine functions, and $\dot{C}(\alpha, q, t)$ and $\dot{S}(\alpha, q, t)$ are their time derivatives. From the expression $C(\alpha, q, t) = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\phi t}P(\alpha, q, t) + \mathrm{e}^{-\mathrm{i}\phi t}P(\alpha, q, -t))$, where $P(\alpha, q, t)$ is a periodic function with a period π and $\phi = \phi(\alpha, q)$ is the so-called characteristic exponent, it can be seen that, if $\mathrm{Im}(\phi) \neq 0$ the solutions exhibit an unstable behavior. Otherwise the solutions are uniformly bounded. We are interested here in a situation in which the second uncoupled forced oscillator (system) is in the stability region and the other (environment) is in the unstable region. From the solutions of the Heisenberg equations of motion and the definition of the upper quantum Lyapunov exponent $\bar{\lambda}$, it is a simple matter to show that $\bar{\lambda} = |\mathrm{Im}(\phi(\alpha_1, q))|$. The connection with the information loss is made through the evaluation of the Schrödinger determinant for the second oscillator, which turns out to be of the form of equation (4.7). In this case, however, the coefficients C_{ab} are periodic functions of time. Moreover, $\bar{\lambda} = |\mathrm{Im}(\phi(\alpha_1, q))|$ and $\mu = \mathrm{i}\phi(\alpha_2, q)$. For an initial Gaussian state, $S(t_n) \sim \ln(C_{20})/2 + \lambda t_n$, where $t_n = 2\pi n$. The coefficients C_{ab} are obtained from the asymptotic behavior of entropy (2.6).

6. Conclusions

In spite of exhibiting a complex, rich dynamical behavior, bipartite open quantum Anosov systems display, in their instability region of parameters, a reduced von Neumann entropy with linear growth determined by their uqLe, a universal behavior for this class of systems, in contrast to QBME. This also permitted us to find a unified description of two opposite regimes, namely, of small and large coupling. It may, however, be argued that (4.7) is *induced* by classical mechanics: the value of the uqLe for the present models equals the classical maximal Floquet exponent, which is positive in the region where the classical parameter $E = \frac{1}{T}\int_0^T f(t)\,\mathrm{d}t$ lies in the instability region (gap) of Hill's equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + f(t)x = 0$ [28], as may abstracted from the previous section. This is a consequence of the fact that the dynamics of the *observables* is classical. The space of *states* is, nevertheless, subject to the laws of quantum mechanics, without classical analogue, even in these cases: for this reason, for the Gaussian initial states, the entropy growth (4.8) is to be interpreted, due to (2.7), as coherence loss, a purely quantum mechanical concept. Indeed, in contrast to the quantum mechanical entropy, the classical entropy is not an adequate quantity in information theory, because it is not even *positive* in general [29, proposition 1]. Negative entropies are due to the fact that the classical density distributions may be concentrated in the regions of phase space $< h$, contradicting the uncertainty principle [30, 31]. Modified 'semiclassical entropies' without this inconvenience have been defined by Wehrl [31], but they incorporate the Hilbert space structure of quantum physics, i.e., of a countable discrete basis of states. Entanglement has indeed been described in terms of this modified entropy [32]. Thus, an explanation by 'classical entanglement' of the phenomena of information loss is not possible in a full classical treatment involving the entropy.

A still different approach is to regard the entanglement in terms of the Wigner function, i.e., in phase space, which illuminates quite different and surprising aspects [33].

We should caution the reader that, even in the framework of (time-dependent) quadratic Hamiltonians, our approach misses an important element which is present in fully chaotic systems: *folding*, which is the characteristic of a compact phase space. Indeed, other papers have reported on the growth of relative entropy in the context of quantized chaotic systems [34, 35]. In contrast to the present treatment they rely, however, on approximations (based on the random matrix theory in [34] and semiclassical analysis in [35]), numerical studies and certain special constructions (the nonunitary step in [35] is constructed to mimic the presence of diffusion) and special concepts (a particularly phase-space measure of complexity was used in [34]). We hope that the discrete Weyl–Wigner formalism developed in [36] may be used to extend our treatment to some quantum chaotic systems.

As a final remark, we mention that other parameters could be invoked for environments characterized by generic chaotic systems: the diffusion coefficient is one of them, which is conjectured to be related to the Lyapunov exponent and the Hausdorff dimension [37].

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References

[1] Schrödinger E 1935 *Proc. Camb. Phil. Soc.* **31** 555
[2] Einstein A, Podolski B and Rosen N 1935 *Phys. Rev.* **47** 777

[3] Bell J S 1964 *Physics* **1** 195
[4] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[5] von Neumann J 1955 *Mathematical Foundations of Quantum Mechanics* (Princeton, NJ: Princeton University Press)
[6] Giulini D, Joos E, Kiefer C, Kupsch J, Stamatescu I O and Zeh D H 2003 *Decoherence and the Appearance of the Classical World in Quantum Theory* (Berlin: Springer)
[7] Caldeira A O and Leggett A J 1983 *Ann. Phys.* **149** 374
[8] Haake F 2001 *Quantum Signature of Chaos* (Berlin: Springer)
[9] Zurek W 2001 *Nature (Lond.)* **412** 712
[10] Alicki R 2002 *Preprint* quant-ph/0205173
[11] Prosen T and Znidaric M 2002 *J. Phys. A: Math. Gen.* **35** 1455
[12] Blume-Kohout R and Zurek W 2003 *Phys. Rev.* A **68** 032104
[13] Zurek W H and Paz J P 1994 *Phys. Rev. Lett.* **72** 2508
[14] Pattanayak A and Brumer P 1997 *Phys. Rev. Lett.* **79** 4131
[15] Pattanayak A 1999 *Phys. Rev. Lett.* **83** 4526
[16] Katok A and Hasselblatt B 1955 *Introduction to the Modern Theory of Dynamical Systems* (Cambridge: Cambridge University Press)
[17] Penrose O 1979 *Rep. Prog. Phys.* **42** 1937
[18] Emch G G, Narnhofer H, Sewell G L and Thirring W 1994 *J. Math. Phys.* **35** 5582
[19] Jauslin H R, Sapin O, Guérin S and Wreszinski W F 2004 *J. Math. Phys.* **45** 4377
[20] Agarwal G S 1971 *Phys. Rev.* A **3** 828
[21] Serafini A *et al* 2004 *J. Phys. B: At. Mol. Opt. Phys.* **37** L21
[22] Magalhães A R B *et al* 2006 *Phys. Scr.* **74** 472
[23] Ozorio de Almeida A M 1988 *Hamiltonian Systems: Chaos and Quantization* (Cambridge: Cambridge University Press)
[24] Sapin O, Jauslin H R and Weigert S 2007 *J. Stat. Phys.* **127** 699
[25] Arnold V 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)
[26] Broer H and Simó C 1998 *Bol. Soc. Bras. Mat.* **29** 253
[27] Casati G and Molinari L 1989 *Prog. Theor. Phys. Suppl.* **98** 287
[28] Hochstadt H 1971 *The Functions of Mathematical Physics* (New York: Wiley)
[29] Robinson D W and Ruelle D 1967 *Commun. Math. Phys.* **5** 288
[30] Wehrl A 1978 *Rev. Mod. Phys.* **50** 221
[31] Wehrl A 1979 *Rep. Math. Phys.* **16** 353
[32] Mintert F and Zyczkowski K 2004 *Phys. Rev.* A **69** 022317
[33] Almeida A M O 2006 *Preprint* quant-ph/0612029
[34] Bandyopadhyay J N and Lakshminarayan A 2004 *Phys. Rev.* E **69** 016201
Lakshminarayan A 2001 *Preprint* quant-ph/0107078
[35] Bianucci P *et al* 2002 *Phys. Rev.* E **65** 046226
[36] Galetti D and Toledo Piza A F R 1995 *Physica* A **214** 207
[37] Gaspard P, Claus I, Gilbert T and Dorfman J R 2001 *Phys. Rev. Lett.* **86** 1506